

## **The Effect of Different Geometries on Percolation in Two Dimensions**

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*Using the model simulation Jared Weddell and Dr. Alan Feinerman have developed, this study found results for the percolation threshold as a function of aspect ratio for rectangle shaped holes and compared these results to Weddell et. al.'s elliptical ones [1]. We developed a slightly different experimental set-up than Weddell, proving that this method for collecting percolation data is repeatable and valid. The percolation threshold for both shapes has been found to exhibit a similar trend - as aspect ratio decreases, percolation threshold increases. At small aspect ratios, when both shapes are almost indistinguishable lines, the percolation threshold results are very similar. For larger aspect ratios, the differences in percolation threshold are much greater. More shapes need to be considered, but these results imply that there exists a point at which the aspect ratio of a shape does influence the percolation threshold.*

## Introduction

Percolation describes the movement of a fluid across a porous material.<sup>1</sup> The phenomenon pervades our everyday life, but it requires complicated measures to quantify.<sup>2,3</sup> Percolation theorist, Peter Kleban writes, "Although percolation is... arguably the simplest model ... the ease of formulation of the model is in the other sense deceptive, tending to conceal its inherent complexity."<sup>3</sup> Percolation as a field has implications to the studies of not only soil physics and geology, but also biology. Agricultural scientists interested in studying the flow of water carrying nutrients through soil would use percolation models. So too, would geologists interested in fluid flow through micro-fractures in rocks.<sup>4</sup> Doctors modeling the diffusion of drugs through the bloodstream have considered fractal and percolation cluster formations.<sup>5</sup>

The specific quantity of interest to us in this study is the percolation threshold,  $p_c$ . It is the smallest pathway across which fluid can flow. In more measurable quantities, it means the critical fraction of area for which a medium can still be electrically or thermally conductive.<sup>6</sup> It is of interest to us because it can relate thermal and electrical conductivity, or a medium's diffusion constant to a single quantity of area through which electrons or a fluid can flow.

In our experiment, we cut pores on a region until there exist no pathways which current could flow across. Through experimental measurements of current across a percolated region and a proxy region, which represents an equivalent amount of area remaining across the first, we can determine a value for  $p_c$ . In this specific study, random networks of 1500 rectangle shaped pores of variable aspect ratios were created. Aspect ratio is the ratio of a rectangle's length to width. Rectangles of aspect ratio 0.0125, a long stick, to 1.000, a perfect square, were cut. We want to look at rectangles especially, because experimental research for the shape has not been done before to our knowledge. We measured how the changing aspect ratio for rectangles affects the percolation threshold, and then compared our results to the previous experimental ellipse results of Weddell et. al.<sup>1</sup>

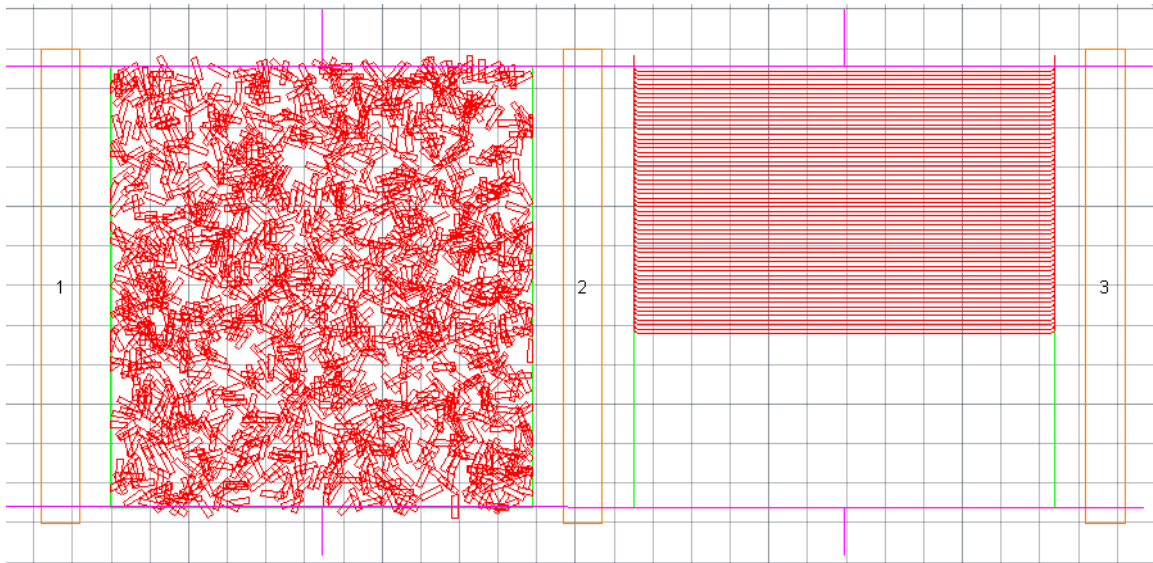
In the next section, we will discuss the framework for our percolated systems and describe, in a more mathematically explicit fashion, how the percolation threshold was calculated. Then, the physical set-up for our electrical model will be described. Lastly, we will present our results for the percolation threshold with respect to changing aspect ratio, discuss the data's relationship to elliptical results, and cite possible sources of error.

## Experimental Design and Procedure

In order to create a percolation pattern, the computing program MATLAB was used. Rectangles of specified aspect ratios were randomly generated and oriented using the program. MATLAB continues to arrange shapes beyond the point at which any current could flow across the percolated square (left square in Fig. 1). The code also allows us to change the number of cuts made. In this study, the

number of cuts was kept constant at roughly 1500 cuts. (According to percolation theory, the number of cuts should not affect the value of the percolation threshold obtained.<sup>6</sup>)

MATLAB outputs a random system of pores to a script file, which the drafting program, AutoCAD, reads. From AutoCAD, we can output a design, for example the one shown in Fig. 1, to our Universal 100 Watt CO<sub>2</sub> laser, which cuts the design into a sheet of Mylar with aluminum coating. The CO<sub>2</sub> laser cuts a few rectangles of a specified aspect ratio on the first square, our percolated region, and across the second, the laser will remove the effective amount of area for each series of cuts (right square in Fig. 1). (Precisely, the effective area square is updated every time 1% or more of the area on the percolated sheet has been decreased.) Thus, this second square reflects the appropriate amount of area remaining across the percolated square.



**Fig. 1:** Sample AutoCAD design for overlapping rectangles of aspect ratio 0.2500. The percolated region is on the left, and the effective area display is on the right.

The percolation threshold, the quantity our study focuses on, is a percentage of area that must be removed before electrical conduction stops. We can determine this critical fraction using the definition of resistance and looking at the change in current across our effective area region. First, in Equations 1 and 2 we use the definition of resistance. They describe the initial and final resistances,  $R_0$  and  $R_r$  respectively, across our effective display area. Below,  $\rho$  is a constant for the resistivity of our conductive sheet,  $L$  is the constant length across which we are measuring current, and  $\tau$  is the constant thickness of our conductive sheet.  $H_0$  is the original height of the effective area square, and  $H_r$  is the height of the area remaining of that same square when the first square has been fully percolated.

$$R_0 = \frac{\rho L}{\tau H_0} \quad (1)$$

$$R_r = \frac{\rho L}{\tau H_r} \quad (2)$$

The percolation threshold is a ratio of area across the effective area side, when no current is flowing across the first square, to the initial area (see Equation 3 below.)

$$p_c = \frac{H_r L}{H_0 L} = \frac{H_r}{H_0} \quad (3)$$

Using Equations 1, 2, and 3, the percolation threshold can be expressed in terms of resistances  $R_0$  and  $R_r$ . In our experiment, a constant voltage was applied as current was measured across both squares. Using Ohm's Law, we can express the percolation threshold in terms of currents  $I_0$  and  $I_r$  (see Equation 4).

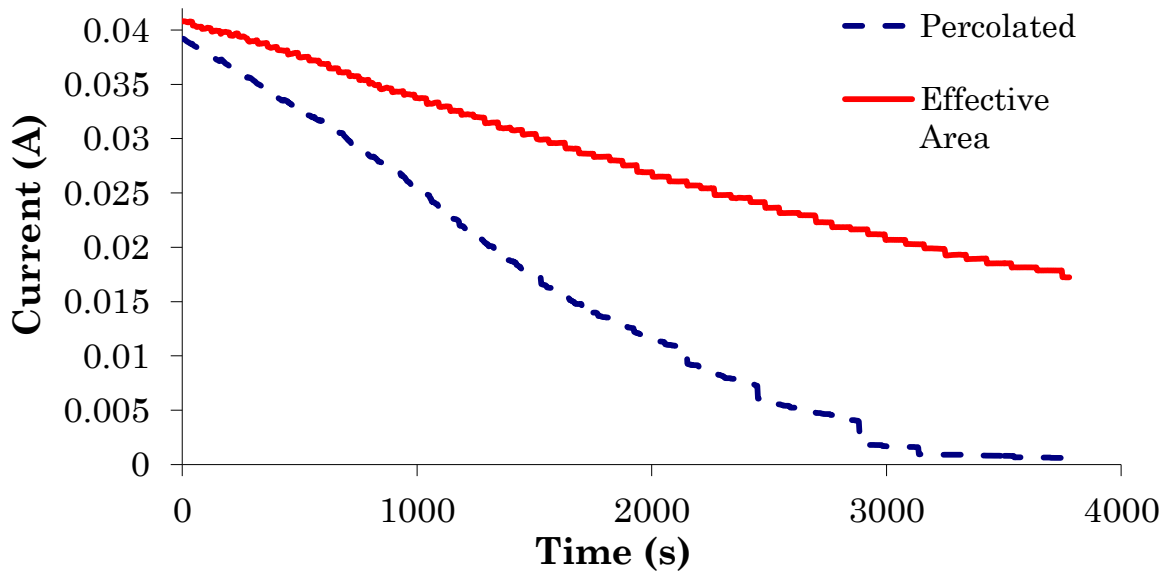
$$p_c = \frac{(\rho L / \tau R_r)}{(\rho L / \tau R_0)} = \frac{R_0}{R_r} = \frac{I_r}{I_0} \quad (4)$$

### Electrical Model

The design of the electrical model used is nearly the same as Weddell's setup.<sup>1</sup> The apparatus consists of an aluminum baseplate, which has a layer of double-sided adhesive sheet and a thin layer of conductive aluminum Mylar placed flat on top. Our model, unlike Weddell's, can fit two runs of data, and features more space on the sides of both the percolated and effective area squares.<sup>1</sup> These edges contribute some resistance, which we take into account for our calculation of the percolation threshold. The conductive sheet is divided into two squares of the same area, which are our percolation and effective area regions. At the same time as the laser is cutting both sheets, the current is measured with alligator clips on two brass rods placed across each square. Not only are the brass rods used to measure the current across both regions, but they also hold the Mylar in place. The brass rods used are secured in place with acrylic fixtures and nylon screws (unlike Weddell's metal screws).<sup>1</sup> In this model, as shapes are cut across the first square, we eliminate possible pathways for electrons to flow. So as we measure current across both squares, the current decreases to zero (resistance becomes infinite) across the percolated square. (This can be seen in Fig. 2, a representative of typical raw data.)

### **Results**

Fig. 2 shows typical results from a single trial. You can see as the laser continues to cut the conductive Mylar, the current across the percolated square reaches zero (dashed), while the current measured across the effective area square (solid) steadily decreases.

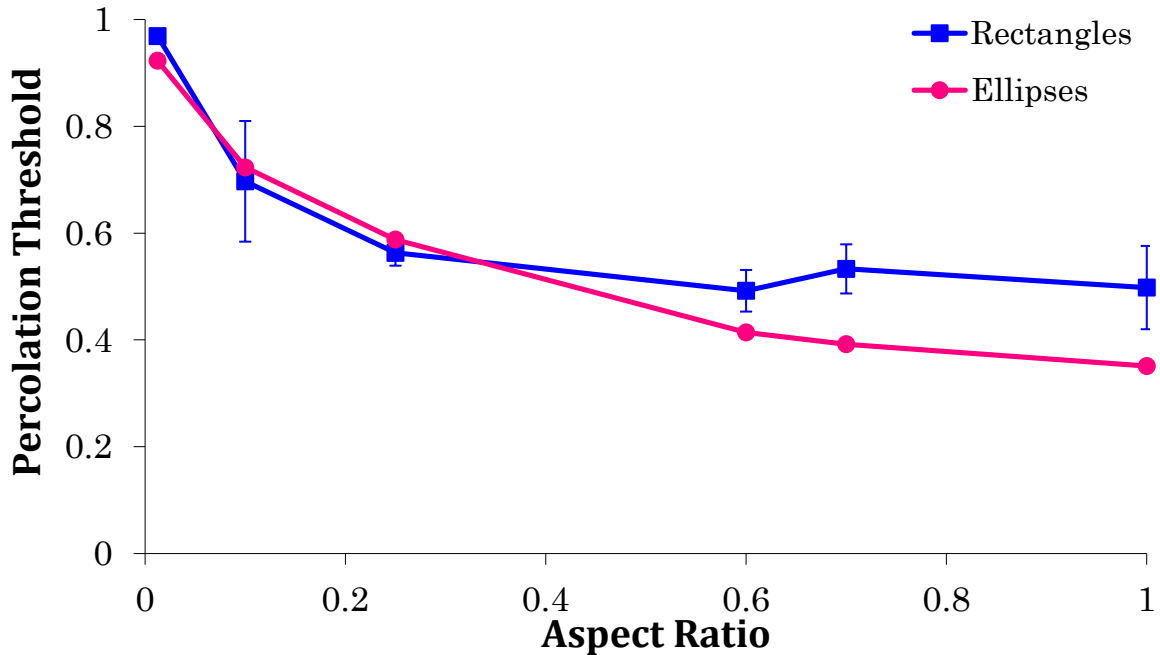


**Fig. 2:** Representative of data collected. As time continues, more area is cut away, and current decreases. Area removed is a non-linear function of time.

Table 1 features the results of our experiment. For each aspect ratio, 3-6 different patterns of roughly 1500 rectangular shapes were cut and the average percolation threshold is listed below. The  $\pm$  value listed is the standard deviation found for the multiple measurements taken at each aspect ratio. The percolation threshold has been calculated using Equation 3 and our experimentally measured values for current. These experimental results for rectangles are compared to elliptical results in Fig. 3 below.

**Table 1:** Results for the percolation thresholds of rectangles of variable aspect ratios, as compared to Weddell's ellipse data. Each rectangle result features  $\pm$  the standard deviation for  $p_c$  at that aspect ratio.

| Aspect ratio | Percolation threshold for rectangles ( $\pm$ STD ) | Percolation threshold for ellipses [6] |
|--------------|--|--|
| 1.0000       | $0.498 \pm 0.078$                                  | 0.351                                  |
| 0.7000       | $0.533 \pm 0.046$                                  | 0.392                                  |
| 0.6000       | $0.492 \pm 0.039$                                  | 0.414                                  |
| 0.2500       | $0.563 \pm 0.024$                                  | 0.588                                  |
| 0.1000       | $0.697 \pm 0.113$                                  | 0.723                                  |
| 0.0125       | $0.969 \pm 0.008$                                  | 0.923                                  |



**Fig. 3:** A plot of percolation threshold versus aspect ratio for rectangles (square markers) and ellipses (circle markers). Error bars for the standard deviation have been included for each rectangle result.

### Discussion

Many control tests were run in order to replicate the results of last summer. We collected 5 runs of data for 1000 circle cuts and found an average  $p_c$  of  $0.34 \pm 0.073$ , close to the theoretical value Xia and Thorpe cite, 0.33, and Weddell's value, 0.36.<sup>7,1</sup> From this result, we can conclude that this experimental method is repeatable and valid.

The process of creating a percolated system is random, so it inherently will produce slight deviations for our results. By creating multiple distinct patterns for shapes of the same aspect ratio, and averaging the percolation threshold for each of those trials, we can try to determine the most appropriate value. For results with initially high standard deviations, more measurements of the threshold were taken to try to more accurately define the value. For example, the standard deviations for our percolation thresholds at aspect ratios 1.0 and 0.1 are relatively large. We could perhaps more accurately determine these results by collecting more data for these aspect ratios. Results show a relatively low value for the percolation threshold at an aspect ratio of 0.6000. This is likely due to a statistical error.

Errors beyond statistical errors are very hard to measure. The error found in Weddell et. al.'s previous experimental result was attributed to an underestimating of the kerf of our laser, where kerf is the width of a laser's cut. A MATLAB code was developed that arranged rows of circles a specified width apart. When no current was measured across the region, we knew the distance between each row was the

same as the kerf of the laser. We have precisely found the kerf to be 114 microns, so we believe we have little error associated with the laser, unlike Weddell's results.<sup>1</sup>

It is seen in Fig. 3, that generally, for rectangle shaped holes, the percolation threshold increases as the aspect ratio decreases. This is a trend that ellipses also exhibit.<sup>1</sup> This result is explained by Weddell as being due to the fact that holes that are nearly lines will still block the current from flowing, but have much smaller area as compared to circles or squares.<sup>1</sup> At small aspect ratios, the percolation thresholds for ellipses and rectangles are close to each other. This makes sense because at an aspect ratio of 0.0125, both rectangles and ellipses are very thin and appear to be nearly the same shape. At larger aspect ratios, the difference between percolation threshold values for rectangles and ellipses is more significant because these shapes are more distinct. This result seems to suggest that there is a value of the aspect ratio at which the shape's geometry seems to uniquely affect the measured percolation threshold. (Our best guess for this value would be somewhere around an aspect ratio of 0.5000.)

## Conclusion

This study verifies that the experimental method Weddell and Feinerman have developed does indeed seem valid. We have repeated circle results and found a  $p_c$ , not only close to Weddell's result, but also consistent with prior published theoretical values. Our results for rectangle shaped holes exhibit a very similar trend as our elliptical results – as the aspect ratio approaches zero, the percolation threshold approaches one. At small aspect ratios, we find our rectangle results in close agreement with elliptical data. The most significant observation we can make looking at our data is that for some critical aspect ratio, the percolation thresholds for rectangles begin to differ from ellipses (around roughly 0.5000). Future work could be done trying to more precisely pinpoint the aspect ratio at which the percolation threshold begins to deviate. Also more work could include looking at different shapes, such as triangles.

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