Percolation Properties of Triangles With Variable Aspect Ratios

Gabriela Calinao Correa
Professor Alan Feinerman

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ABSTRACT

Percolation is the study of random networks and their nature of connectivity. An exponential relationship between the percolation threshold and the aspect ratio of a shape is observed within this paper. Matlab simulations and a novel experimental method is utilized to determine the percolation threshold across a conductive sheet as triangles of variable aspect ratios are cut with a Universal Laser System X2-600 100 Watt CO_2 laser. Resistance is measured as a voltage of 100mV is applied across the conductive sheet of aluminum coated Mylar film. This data is then analyzed to yield the percolation threshold relating to a triangle with a specified aspect ratio. We establish an equilateral triangle to have an aspect ratio of one.

INTRODUCTION

Percolation Theory studies random networks and their nature of connectivity [5]. Envision a compressed cube of particles, blocking off a source of water. One particle is randomly removed at a time, percolating the cube. The percolation threshold P_C is met when a path opens up such that the water is permitted to flow.

We investigate the percolating network of triangles of various aspect ratios being cut across a conductive sheet, with electrical current as the medium flowing across the network. Previously, the relationship of an ellipse's aspect ratio to the percolation threshold has been established both experimentally and theoretically as $p_c = e^{-\pi ab n_c}$ where n_c is the hole density per unit volume at the percolation threshold, and $\frac{b}{a}$ is the aspect ratio of the ellipse [1][2]. The a and b depict the upper and lower range of the ellipse's diameters.

Triangles also have an upper and lower range of diameters. Given the same range, ellipses and triangles have a different distribution of diameters, given their different shapes. In addition, aspect ratios are defined differently. For an ellipse, an aspect ratio of 1 indicates a circle. For a triangle, an aspect ratio of 1 is used to represent and equilateral triangle. Formulaically, an aspect ratio R_T for triangles of height h and base b is defined as follows:

$$R_T = \frac{h}{b\sqrt{3}} \qquad (1)$$

This differs from the simple $\frac{b}{a}$ indicating the aspect ratio for both rectangles and ellipses.

METHODS

Overview

To generate data, a Matlab program is utilized to generate an AutoCAD script file, which is then used to guide a Universal Laser System X2-600 100 Watt CO_2 laser. The laser cuts shapes out of a sheet of aluminum coated Mylar film, and current is measured across the sheet as each cut is made. Over time, cuts are made within the percolation area. When current can no longer flow across the sheet, it is said the percolation threshold has been met.

Matlab Algorithm

The Matlab code geometrically defines the AutoCAD file. The benefit to utilizing this method is that pseudo-random figures may be generated, and all drawings may be precisely automated. The percolation area is defined with width W and height H, forming a W by H rectangle. Within this rectangle, triangles with a defined aspect ratio are generated [2]. Formula (1) indicates the relationship between base b and height h to the aspect ratio R_T of a triangle, which defines the aspect ratio of an equilateral triangle to be one. This formula can then be combined with the formula for triangle area $A = \frac{b\,h}{2}$ resulting in a formula for base and height of a triangle, given a fixed area A and an aspect ratio R_T.

$$\mathrm{b} = \sqrt{\frac{2\mathrm{A}}{\mathrm{R}\sqrt{3}}} \qquad (2)$$

$$\mathrm{h} = \mathrm{bR}\sqrt{3} \qquad (3)$$

The vector depicting the vertices of the oriented triangle are centered around uniformly distributed pseudo-random x and y coordinates. This vector is then put through a rotation matrix with a pseudo-randomly generated angle, theta. The result is a list of vertices depicts a triangle with random location and orientation.

Next, the triangles are drawn within the defined percolation area. If the triangle does not intersect the edges of the percolation area, it is simply drawn. However, if an intersection exists, the triangle must be drawn as a polygon including the edges of the percolation area. The intersection points are determined with knowledge of the two vertices they are in between. The first intersection point is added to a vector that defines the polygon's coordinates. Next, the vertex surrounding the first intersection that is inside the box is added. If two points are in the box, then the second point within the box is added. Finally the second intersection point is added. If the triangle is on a corner of the box, the corner point is added at the end.

As each triangle is drawn, the loss of area is calculated. Once area in the percolation sheet is reduced by a threshold value of 0.5 percent, a strip of equivalent area is removed from another part of the conductive sheet. The loss of area is determined by a matrix of nodes, to represent whether or not that area has been previously removed. The matrix of nodes is further used to label clusters; when one cluster contains a node from each edge of the percolation area, the percolation threshold is met.

Setup

The experimental setup utilizes electrical properties of a thin conductive sheet to determine if the percolation threshold has been met. Each setup holds two samples to be percolated. The base providing structure to the setup consists of an aluminum base, with thirty-six holes drilled and threaded for 4-40 screws. The aluminum base is sandblasted, to remove its inherently reflective surface; this protects the laser and its lenses from damage. Nylon screws are utilized to prevent a short circuit from the sample to the aluminum base. Atop the aluminum base, two adhesive strips of double-sided masking tape are placed between the screw rows. To this, paper is attached to further protect the aluminum base from laser damage, so that it may be reused. After two layers of standard printing paper, a conductive sheet of aluminum coated Mylar film is attached. This conductive sheet is previously adhered to a strip of the double-sided masking tape carefully such that few surface imperfections exist.

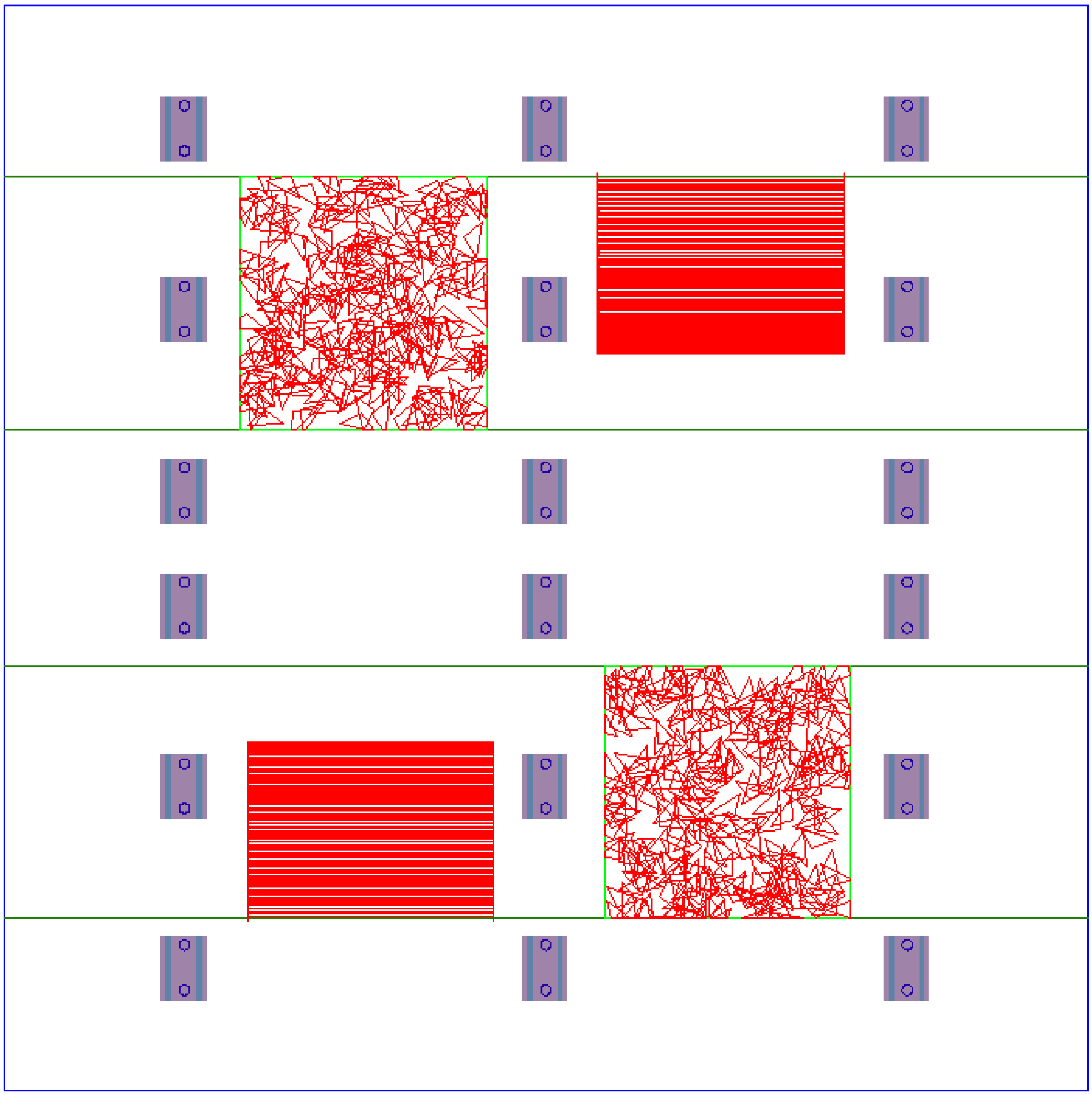

Figure 1: AutoCAD representation of the experimental setup. The exterior blue border indicates the area occupied by the sandblasted aluminum base. The green lines show the upper and lower boundaries of the two conductive strips, each holding a sample. These green lines are also utilized to guide the laser to trim the aluminum coated Mylar film, such that no path exists above or below the percolation area for

current to flow. The purple rectangles depict the acrylic fasteners, each with two holes for screws. The cyan stripes along each edge of the acrylic fasteners are rasterized areas, to provide an indent for the brass rods. The triangles in this figure are enlarged for illustration.

After the base has been adhered to both paper and conductive sheet, the laser is used to trim the aluminum coated Mylar film. To cut the Mylar film, a speed of 100% (appx. 0.5 m/s) and a power of 40% (i.e. 40 Watts) is used. This eliminates any possible path above or below the percolation area for current to flow. Next, acrylic fasteners held by two screws each are placed atop two brass rods. The brass rods surround each area across which a voltage is applied, and resistance is measured.

Each run consists of an area with triangles, and an area with strips. Resistance is measured across each area. As a significant amount of area is removed from the triangle area, an equivalent amount of area is removed from the side with strips. This allows the removal of triangles over time to relate to the amount of area lost. The point of time where current can no longer flow across the triangularly cut area effectively coincides with the time the stripped area is a certain resistance. Resistance across the stripped area may be related to the amount of area removed, by utilizing a formula for sheet resistance.

RESULTS

Data is gathered from both Matlab simulations and the experimental setup previously described. A sample of the resistance measured over time can be seen in Figure 2. Figure 3 illustrates the triangular area being cut. Initially, the resistance is finite (typically ~3Ω), indicating that there is a path for current to flow across. When the resistance is no longer finite and current may no longer flow across the sheet, the resistance values become noise. The point in time at which this occurs coincides with the striped side, whose resistance over time is graphed in Figure 4. The resistance plateauing in Figure 4 simply indicates that no more area is being removed, despite continuous measurements. As can be seen, the resistance of the striped area smoothly increases. This easily relates the amount of area removed, and therefore the percolation threshold P_C.

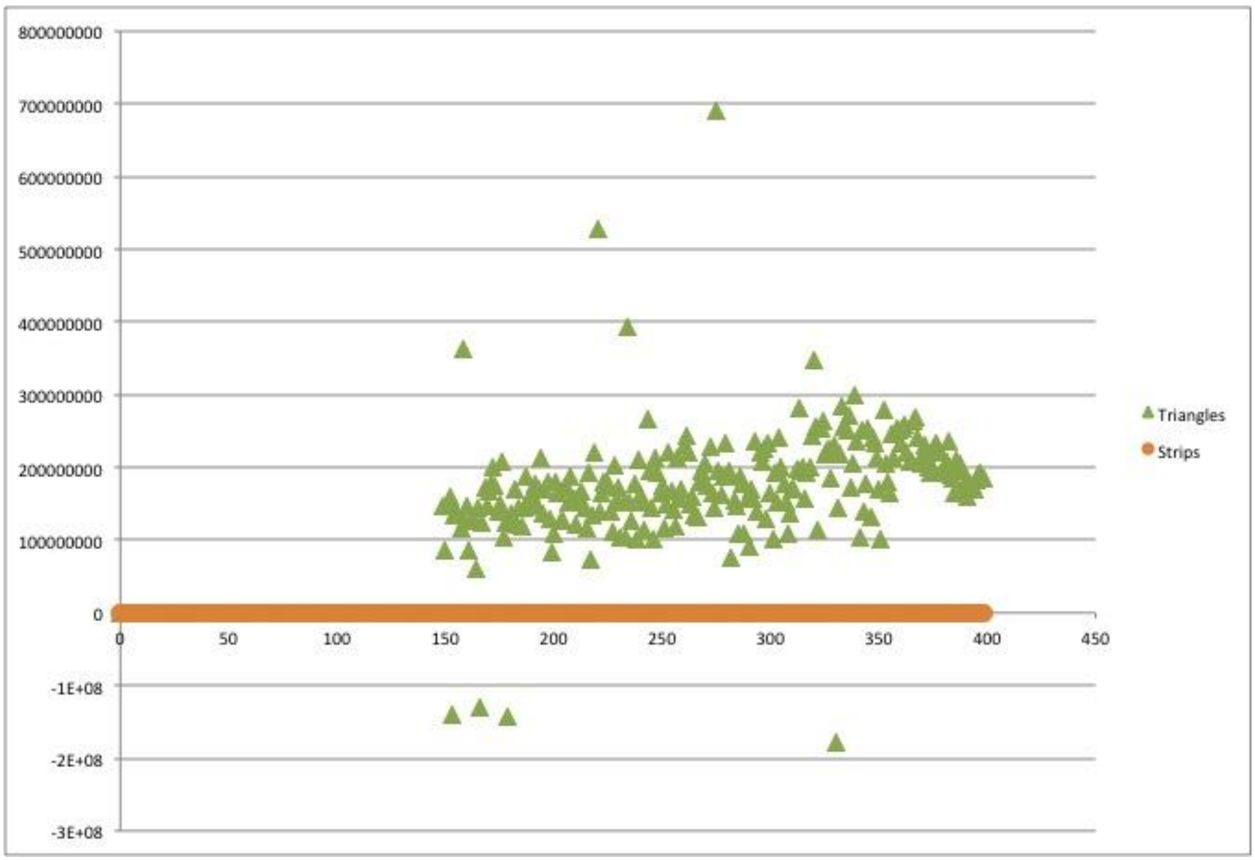

Figure 2: Sample of data gathered across a percolating system.

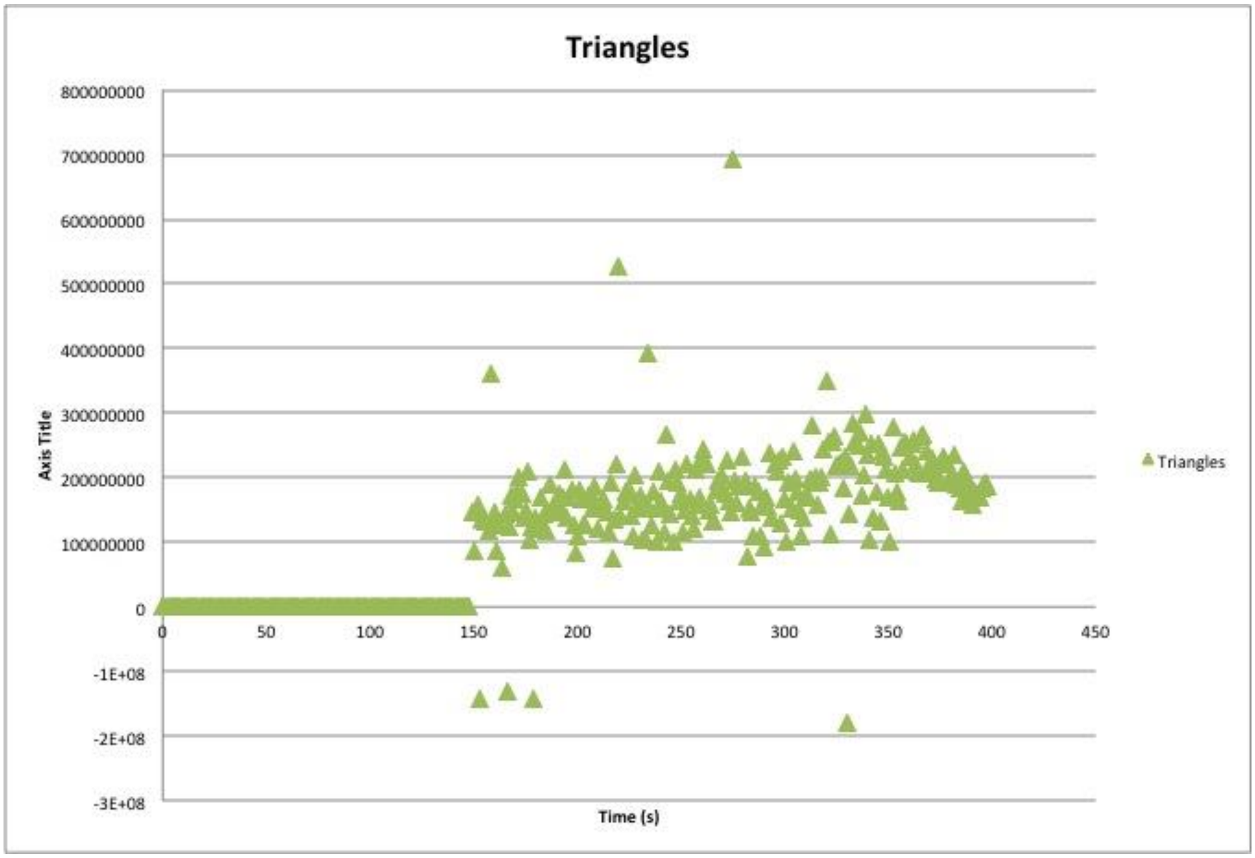

Figure 3: Sample of data gathered across a percolating system.

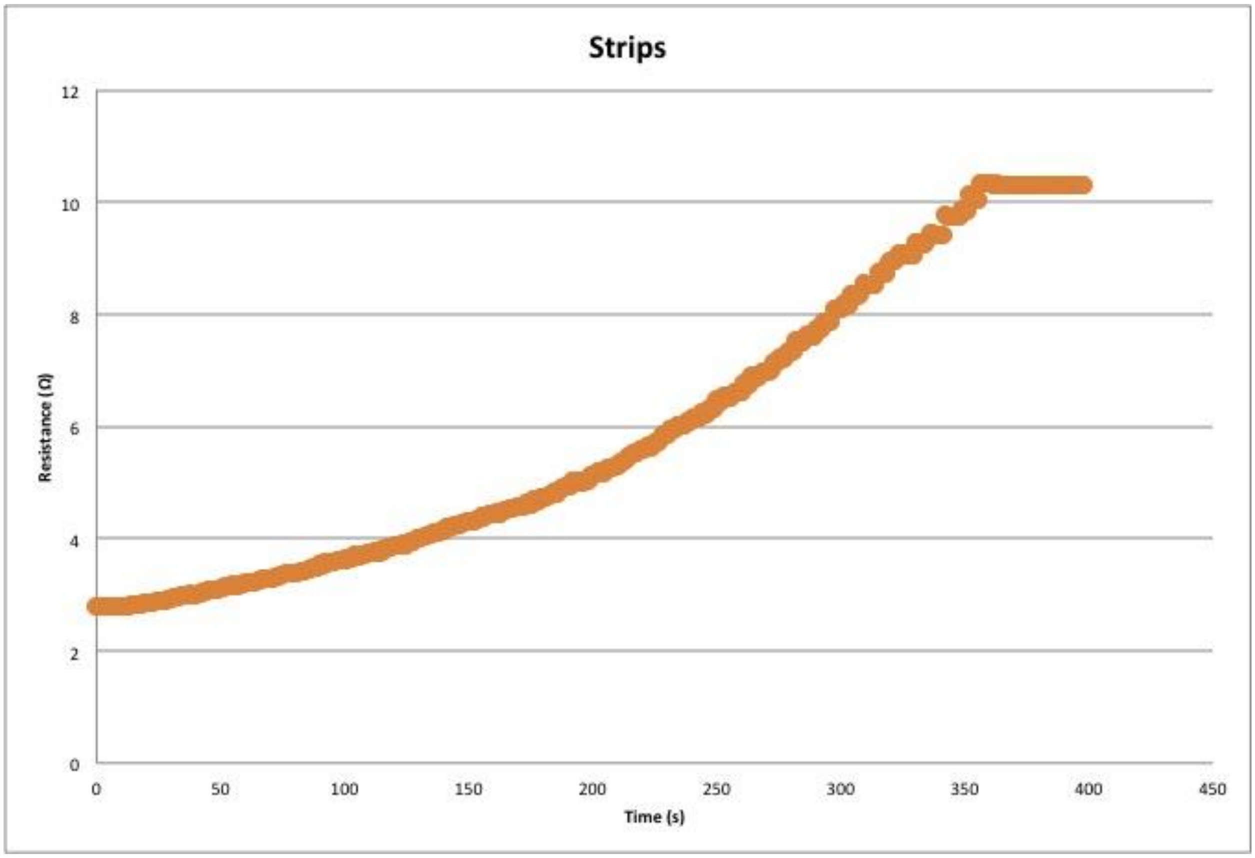

Figure 4: Sample of data gathered across a percolating system.

After gathering the percolation thresholds for various aspect ratios from these measurements, they are plotted in Figure 5 alongside the data gathered from simulations. The blue indicates the simulated data, while the red asterisks indicate experimental data. There are six relevant experimental data points; the remainder of the data was determined to be unusable due to experimental error.

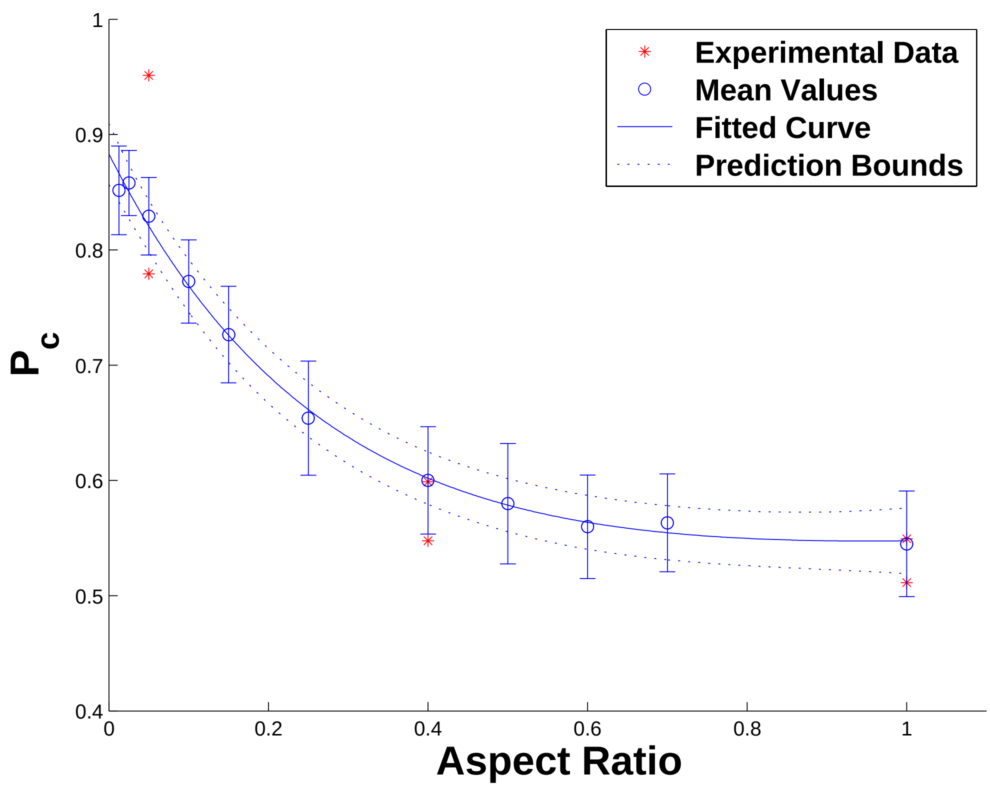

Figure 5: The data here is a compilation of data relating the percolation threshold, P_c and the Aspect ratio of a Triangle, where an equilateral triangle sets the base aspect ratio of 1. In blue, data simulated by Matlab is shown. For each mean value point, a hundred data values are used. The error bars are the standard deviation of these values. The fitted exponential curve is similar to previously established data, and the experimental data points gathered appear to show a similar trend.

Experimental error was determined to be largely due to the areas of the triangles used. Figure 6 consists of two simulated runs of data, named A and B. Column A consists of the means acquired after a hundred runs of data with triangles of area of 11.1613μm, while Column B consists of the means acquired after ten runs of data with triangles with an area of 1.2903μm. The standard deviations of B are significantly smaller than that of A. This indicates that smaller triangles do indeed result in less error.

Aspect Ratio	Ellipses	Rectangles	A	STD (A)	B	STD (B)	C	STD(C)
1.0000	0.351	0.498	0.5450	0.0435	0.5747	0.0202	0.5303	0.0190
0.7000	0.392	0.533	0.5632	0.0421				
0.6000	0.414	0.492	0.5599	0.0451				
0.5000			0.5798	0.0502				
0.4000			0.6001	0.0477	0.6334	0.0290	0.5732	0.0256
0.2500	0.588	0.563	0.6541	0.0507				
0.1500			0.7265	0.0420				
0.1000	0.723	0.697	0.7727	0.0366				
0.0500			0.8292	0.0333	0.7144	0.0206	0.8653	0.0860
0.0250			0.8581	0.0278				
0.0125	0.923	0.969	0.8516	0.0386				

Figure 6: Data of percolation thresholds in relation to Aspect Ratio, with standard deviations. Data for Ellipses [2] and Rectangles [4] was gathered by Feinerman et al, while column A consists of the means from a hundred simulated runs of triangles with area of 11.1613μm, and column B consists of the means from ten simulated runs of triangles with area of 1.2903μm. Column C consists of means from two experimental runs of data.

Column C of Figure 6 consists of means from two experimental runs of data, where triangles had an area of 2.5161μm. The trend gathered through the theoretical and the experimental runs follow the similar exponential relationships. With additional data, it is expected that the relationship could be established to be identical.

DISCUSSION

The exponential relationship observed in the final data has previously been observed [1], so it is expected that additional data and analysis could render a definitive formula relating geometric features to the percolation threshold across a network, percolated with various shapes. This may relate to a probability of possible length cut across the primary orthogonal directions of a sample. Shapes with high aspect ratios tend to be like circles or oriented squares in that they provide a relatively consistent length to be removed in these primary directions, whereas shapes with low aspect ratios tend to behave more like line segments when percolating across a network.

Possible error, apart from error resulting from triangle area as previously mentioned, may have originated from various different sources. The random nature of the system introduced some error; truly random values were not utilized, but instead evenly distributed pseudorandom values generated by Matlab. The kerf size of the laser cut introduced additional error in experimental data. The kerf of the laser varied with different geometric figures, and therefore the area removed by the kerf cut was not properly determined per aspect ratio. Lastly, contact resistance due to surface imperfections and contaminants likely introduced some error: The initial resistances across samples were not consistent, indicating that contact resistance may have been significant. To prevent this in the future, alternative methods for measuring sheet resistance—such as a four-point probe—could be utilized to measure the sheet resistance across the striped area of the sample.

CONCLUSIONS

Further work should be done to establish a concrete formula depicting the exponential relationship between the percolation threshold and different geometries. This geometric difference is likely what causes the variations in the relationships between the percolation thresholds and aspect ratios, for varying shapes. Future geometric exploration could be to double these triangles with varying aspect ratios, e.g. double an equilateral triangle into an equilateral rhombus.

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